The collocation method for mixed boundary value problems on domains with curved polygonal boundaries

J. Elschner [*] Y. Jeon [†] I. H. Sloan [‡] E. P. Stephan [§¶]

Abstract

We consider an indirect boundary integral equation formulation for the mixed Dirichlet-Neumann boundary value problem for the Laplace equation on a plane domain with a polygonal boundary. The resulting system of integral equations is solved by a collocation method which uses a mesh grading transformation and a cosine approximating space. The mesh grading transformation method yields fast convergence of the collocation solution by smoothing the singularities of the exact solution. A complete stability and solvability analysis of the transformed integral equations is given by use of a Mellin transform technique, in a setting in which each arc of the polygon has associated with it a periodic Sobolev space.

1 Introduction

Consider the mixed Dirichlet-Neumann boundary value problem for the Laplacian in a simply connected region Ω with piecewise-smooth boundary $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}$: For given f on Γ_D, g on Γ_N, find u in Ω such that

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \Gamma_D, \end{aligned} \tag{1.1}$$

[*] Weierstrass Institute of Applied Analysis and Stochastics, Berlin, Germany

[†] Department of Mathematics, A Jou University, Su Won, Korea

[‡] School of Mathematics, University of New South Wales, Sydney, 2052, Australia

[§] Institut für Angewandte Mathematik, Universität Hannover, Germany

[¶] This research is supported by Australian Research Council

$$\frac{\partial u}{\partial n} = g \qquad \text{on} \quad \Gamma_N,$$

where $\frac{\partial u}{\partial n}$ denotes the derivative of u with respect to the outward normal vector n.

We use the single layer potential for the representation of u,

$$u(P) = -\frac{1}{\pi}\int_\Gamma \log|P-Q|z(Q)dS_Q, \quad P \in \Omega, \tag{1.2}$$

where $|P-Q|$ is the Euclidean distance between P and Q, and dS_Q the element of arc length. From the well known jump condition for the normal derivative of the single layer potential at the boundary, we then have the following boundary integral equations:

$$\begin{aligned} -\frac{1}{\pi}\int_\Gamma \log|P-Q|z(Q)dS_Q &= f(P), \qquad P \in \Gamma_D, \\ z(P) - \frac{1}{\pi}\int_\Gamma \frac{\partial \log|P-Q|}{\partial n_P} z(Q)dS_Q &= g(P), \qquad P \in \Gamma_N, \end{aligned} \tag{1.3}$$

where the density function z is sought on Γ. Throughout the paper we make the following assumption.

(A1) Equation (1.3) with $f = g = 0$ has in $L_p(\Gamma)$ a unique solution $z \equiv 0$ for any $p > 1$.

Defining $z_D := z|_{\Gamma_D}$ and $z_N := z|_{\Gamma_N}$, (1.3) can be rewritten as a 2×2 matrix integral equation system, where z_D and z_N are sought:

$$\begin{aligned} &-\frac{1}{\pi}\int_{\Gamma_D} \log|P-Q|z_D(Q)dS_Q - \frac{1}{\pi}\int_{\Gamma_N} \log|P-Q|z_N(Q)dS_Q = f(P), \quad P \in \Gamma_D, \\ &-\frac{1}{\pi}\int_{\Gamma_D} \frac{\partial \log|P-Q|}{\partial n_P} z_D(Q)dS_Q + z_N(P) - \frac{1}{\pi}\int_{\Gamma_N} \frac{\partial \log|P-Q|}{\partial n_P} z_N(Q)dS_Q = g(P), \quad P \in \Gamma_N. \end{aligned} \tag{1.4}$$

Even for smooth boundary data f, g, the solutions z_D and z_N may not be smooth. Let $\{P_0,\ P_1\}$ be the interface points (i.e., $P_i \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$, $i = 0, 1$). Let us assume the polygon Γ forms an interior angle ω_i at P_i. Then by [3],

$$u(P) = C(\theta)r^{\pi/2\omega_i} + \text{ smoother terms}, \quad P \in \Omega, \tag{1.5}$$

where (r, θ) are the polar coordinates centered at P_i. We may use (1.2) to define a potential not only in the interior region Ω but also in the exterior domain $\mathbb{R}^2 \backslash \overline{\Omega}$. Then the single layer density z is the difference between the normal derivatives of the solution of (1.1) and of u in the exterior domain $\mathbb{R}^2 \backslash \overline{\Omega}$. Thus we have

$$z(P) = Cr^{s_i} + \text{ smoother terms}, \quad s_i = \min\{\frac{\pi}{2\omega_i}, \frac{\pi}{2(2\pi - \omega_i)}\} - 1, \quad P \in \Gamma \tag{1.6}$$

near P_i. Thus z_D and z_N have this behaviour near P_i, possibly with different constants.

For integral equations with solutions having weaker singularities than in (1.6), the mesh grading transformation method has often been applied to obtain a rapidly convergent numerical method [6], [9], [10], [11]. In the following we use a slightly different form of mesh grading analysis, and apply it to the mixed boundary value problem. The idea of the mesh grading transformation is this: if we make a mesh grading $\alpha(x) \approx Cx^q$ near P_i, then instead of z, with the behaviour seen in (1.6), we have to consider

$$\tilde{z}(x) := z(\alpha(x))\alpha'(x) = Cx^{q(1+s_i)-1} + \text{ smoother terms.} \tag{1.7}$$

Now $\tilde{z}(x)$ is smooth for large q, and $\tilde{z}(x)$ can be approximated by an evenly spaced high order spline or a trigonometric function. Moreover, without a mesh grading transformation, the analysis of (1.4) is only possible in a weighted L_2 space or in a Sobolev space of negative order (e.g. $H^{-1/2}$) because of the regularity result (1.6). With a mesh grading transformation, an analysis in the L_2 space is possible.

In this paper we assume for simplicity that Γ_D and Γ_N are smooth arcs. (In the analysis we shall make the stronger assumption, that each arc is straight in some neighbourhood of the corners. This is believed to be an inessential restriction.) The restrictions of $\tilde{z}$ to Γ_D and Γ_N are each approximated by a trigonometric cosine function, with the approximation determined at equally spaced points with respect to the parameter x on each arc. (For a polygon Γ with more than two corners the mesh grading transformation would be carried out for each corner, and the restriction of $\tilde{z}$ to each smooth arc expressed by a different cosine series.)

The analysis has a feature that seems to us unusual, and that perhaps will be useful for other problems. It is that to each smooth arc (after parametrisation as above) we associate a separate periodic Sobolev space. The periodic setting is obtained by extending a function on a given arc (after parametrisation) to twice the natural range of the variable x, by requiring the function to be even about each endpoint. This is an approach which has proved useful in the past for single open arcs (see [13]), and indeed there is a sense in which our first approximation is to treat each arc (after the mesh grading transformation) as an isolated arc.

In working through the analysis, it is important not to be misled into thinking of the above-mentioned extension to an even function as carrying a function defined on one arc of the polygon across to an adjacent arc: rather, the extension to a periodic function carries the parametrisation function $\alpha(x)$ (and hence also every function of $\alpha(x)$) back along the *same* arc. Pictorially, it is useful to think of each arc of the polygon as in some sense a flattened and deformed circle. (The authors understand well the seductiveness of that false view, having often fallen into the trap themselves.)

The paper is organised in the following way. In §2, we introduce the mesh grading transformation, and the mid-point cosine collocation method for the transformed equation is defined. In §4, some preliminary mathematical results regarding the Hilbert transform, a collocation projection on even periodic functions, and the Mellin transform are introduced. The collocation projection is the mid-point collocation, which overcomes an unsymmetric feature of the collocation projection introduced in [1]. In §5, a complete ellipticity and solvability analysis for the mesh-grading-transformed equations arising from (1.4) is given in the L_2 space. In §6 an error analysis for the mid-point collocation method is given.

2 A numerical method

Let us first consider a piecewise-smooth parameterisation $\tilde{\alpha} : [0,2] \to \Gamma$ such that on each smooth arc $|\tilde{\alpha}'|$ is bounded above and below by positive constants, and

$$\tilde{\alpha}([0,1]) \equiv \Gamma_D, \quad \text{and} \quad \tilde{\alpha}([1,2]) \equiv \Gamma_N.$$

Let us consider a mesh grading transformation γ such that, for some ϵ satisfying $0 < \epsilon < 1/2$ and some $q \geq 1$,

$$\gamma(x) = \begin{cases} x^q, & 0 \leq x \leq \epsilon \\ 1-(1-x)^q, & 1-\epsilon \leq x \leq 1. \end{cases} \tag{2.1}$$

The parameter q is the order of the mesh grading. For an example of a good mesh grading transformation, see [11]. Then we consider a new mesh graded parameterisation,

$$\alpha(x) := \begin{cases} \tilde{\alpha}(\gamma(x)), & 0 \leq x \leq 1 \\ \tilde{\alpha}(1+\gamma(x-1)), & 1 \leq x \leq 2. \end{cases} \tag{2.2}$$

We now define

$$\tilde{z}(x) = z(\alpha(x))|\alpha'(x)|, \tag{2.3}$$

and take

$$\begin{aligned} z_1(x) &= \tilde{z}(x), \quad 0 \leq x \leq 1, \\ z_2(x) &= \tilde{z}(x), \quad 1 \leq x \leq 2, \end{aligned} \tag{2.4}$$

so that z_1 and z_2 correspond to the unknown functions on Γ_D and Γ_N respectively. Substituting $P = \alpha(x)$ and $Q = \alpha(y)$, and multiplying the second equation of (1.4) by $|\alpha'(x)|$, we obtain

$$-\frac{1}{\pi}\int_0^1 \log|\alpha(x)-\alpha(y)|z_1(y)dy - \frac{1}{\pi}\int_1^2 \log|\alpha(x)-\alpha(y)|z_2(y)dy = f(x), \quad 0 \leq x \leq 1, \tag{2.5}$$

and

$$\begin{aligned} &-\frac{1}{\pi}\int_0^1 \frac{|\alpha'(x)|(\alpha(x)-\alpha(y), n_x)}{|\alpha(x)-\alpha(y)|^2} z_1(y)dy \\ &\qquad +z_2(x) - \frac{1}{\pi}\int_1^2 \frac{|\alpha'(x)|(\alpha(x)-\alpha(y), n_x)}{|\alpha(x)-\alpha(y)|^2} z_2(y)dy = g(x), \quad 1 \leq x \leq 2, \end{aligned} \tag{2.6}$$

where $f(x) := f(\alpha(x))$, $g(x) := g(\alpha(x))|\alpha'(x)|$, $n_x := n_{\alpha(x)}$ and $(\cdot,\cdot)$ denotes the Euclidean inner product in $\mathbb{R}^2$.

The numerical method is simply to approximate z_1 and z_2 by

$$z_j^h(x) = \sum_{l=0}^{N-1} a_{jl}\cos(\pi l x), \quad j = 1,2, \tag{2.7}$$

and then to collocate equations (2.5) at the 'midpoints' $kh + h/2$ for $0 \leq k \leq N-1$, and equation (2.6) at the points $kh + h/2$ for $N \leq k \leq 2N-1$, where $h := 1/N$.

3 The periodic function space setting

As indicated in the introduction, the first step in the analysis is to introduce a periodic function space setting, in which each arc has associated with it its own periodic Sobolev space. The total function space in which the problem is analysed is then the product of these spaces, with as many spaces in the product as there are arcs (two in the present analysis).

Appropriate Sobolev spaces will be defined in the next section. Here we rewrite the boundary integral equation (2.6) so that it has an appropriate periodic structure.

Recall that the parametrisation function α, defined by (2.2), has values on Γ_D for $0 \le x \le 1$, and values on Γ_N for $1 \le x \le 2$. Let us define the corresponding 2-periodic functions:

$$\alpha_1(x) := \begin{cases} \alpha(x), & 0 \le x \le 1, \\ \alpha(-x), & -1 \le x \le 0, \end{cases} \tag{3.1}$$

$$\alpha_2(x) := \begin{cases} \alpha(x), & 1 \le x \le 2, \\ \alpha(2-x), & 0 \le x \le 1, \end{cases} \tag{3.2}$$

together with

$$\alpha_j(x) = \alpha_j(x+2), \quad j = 1, 2. \tag{3.3}$$

Thus α_1 is the transformation function corresponding to Γ_D, and α_2 the transformation function corresponding to Γ_N. (We would have to define further functions $\alpha_3, \ldots$ if Γ contained further arcs.) Both α_1 and α_2 are even and 2-periodic. (The reader might find it helpful to observe that an even 2-periodic function F is necessarily even about each integer n, since $F(n+x) = F(-n+x) = F(n-x)$.)

In a similar way we extend z_1 and z_2 (the parts of the solution corresponding to Γ_D and Γ_N respectively) to be even 2-periodic functions:

$$\begin{aligned} z_1(x) &= z_1(-x), \quad -1 \le x \le 0, && (3.4) \\ z_2(x) &= z_2(2-x), \quad 0 \le x \le 1, && (3.5) \\ z_j(x) &= z_j(x+2), \quad j = 1, 2. && (3.6) \end{aligned}$$

Then (2.5) and (2.6) can be written as

$$-\frac{1}{\pi}\int_0^1 \log|\alpha_1(x) - \alpha_1(y)| z_1(y)dy - \frac{1}{\pi}\int_1^2 \log|\alpha_1(x) - \alpha_2(y)| z_2(y)dy = f(x), \quad x \in \mathbb{R}, \tag{3.7}$$

$$\begin{aligned} &-\frac{1}{\pi}\int_0^1 \frac{|\alpha_2'(x)|(\alpha_2(x) - \alpha_1(y), n_x)}{|\alpha_2(x) - \alpha_1(y)|^2} z_1(y)dy \\ &\quad + z_2(x) - \frac{1}{\pi}\int_1^2 \frac{|\alpha_2'(x)|(\alpha_2(x) - \alpha_2(y), n_x)}{|\alpha_2(x) - \alpha_2(y)|^2} z_2(y)dy = g(x), \quad x \in \mathbb{R}. \end{aligned} \tag{3.8}$$

Note that the integrals, here and generally in this paper, extend over only half of the period. Further, to avoid unnecessary confusion we have left the intervals of integration as the 'natural' intervals occurring in (2.5) and (2.6). Thus the periodic extensions of each of our solution functions z_1 and z_2 have little effect on the appearance of the equations, while allowing us later a simplified analysis that is only possible in periodic spaces.

Let E denote temporarily the space of 2-periodic, even, complexed-valued measurable functions without regard to smoothness. Then we may define operators $\mathcal{V}_{11}$, $\mathcal{V}_{12}$, $\mathcal{K}_{21}$, $\mathcal{K}_{22}$,

$$\mathcal{V}_{11} z_1(x) = -\frac{1}{\pi} \int_0^1 \log |\alpha_1(x) - \alpha_1(y)| z_1(y) dy, \tag{3.9}$$

$$\mathcal{V}_{12} z_2(x) = -\frac{1}{\pi} \int_1^2 \log |\alpha_1(x) - \alpha_2(y)| z_2(y) dy, \tag{3.10}$$

$$\mathcal{K}_{21} z_1(x) = -\frac{1}{\pi} \int_0^1 \frac{|\alpha_2'(x)|(\alpha_2(x) - \alpha_1(y), n_x)}{|\alpha_2(x) - \alpha_1(y)|^2} z_1(y) dy, \tag{3.11}$$

$$\mathcal{K}_{22} z_2(x) = -\frac{1}{\pi} \int_1^2 \frac{|\alpha_2'(x)|(\alpha_2(x) - \alpha_2(y), n_x)}{|\alpha_2(x) - \alpha_2(y)|^2} z_2(y) dy, \tag{3.12}$$

each of which manifestly maps E to E, and then write our boundary integral equations as

$$\mathbf{B}\,\mathbf{z} = \begin{bmatrix} \mathcal{V}_{11} & \mathcal{V}_{12} \\ \mathcal{K}_{21} & I + \mathcal{K}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{3.13}$$

with $\mathbf{B}$ an operator from $E \times E$ to $E \times E$.

The collocation equations can be written in terms of the operators $\mathcal{V}_{11}$, $\mathcal{V}_{12}$, $\mathcal{K}_{21}$, $\mathcal{K}_{22}$ as

$$(\mathcal{V}_{11} z_1 + \mathcal{V}_{12} z_2)(kh + h/2) = f(kh + h/2), \quad k = 0, \ldots, N-1, \tag{3.14}$$

$$(\mathcal{K}_{21} z_1 + z_2 + \mathcal{K}_{22} z_2)(kh + h/2) = g(kh + h/2), \quad k = N, \ldots, 2N-1. \tag{3.15}$$

4 Spaces and mapping properties

4.1 Sobolev spaces and key operators

Let H^s, $s \in \mathbb{R}$, be the Sobolev space of 2-periodic functions with norm

$$\|f\|_s^2 = \sum_{m \in \mathbb{Z}} \max\{1, |m|\}^{2s} |\hat{f}(m)|^2, \tag{4.1}$$

where

$$\hat{f}(m) = \frac{1}{2} \int_{-1}^1 f(x) e^{-i\pi m x} dx, \tag{4.2}$$

so that

$$f(x) \sim \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{i\pi m x}. \tag{4.3}$$

Following [13], an important role will be played by H_e^s, the subspace of even 2-periodic functions. Similarly, H_o^s denotes the subspace of odd 2-periodic functions, so that

$$H^s = H_e^s \oplus H_o^s, \tag{4.4}$$

expressing the fact that $u \in H^s$ can be written uniquely in the form $u = u_e + u_o$, with $u_e \in H^s_e$ and $u_o \in H^s_o$.

Now let $\mathcal{H}$ be the well-known Hilbert transform on H^s, defined by the principal value integral

$$\begin{aligned} \mathcal{H}u(x) &= -\frac{1}{2}\mathrm{pv}\int_{-1}^{1} \cot(\frac{\pi}{2}(x-y))u(y)dy \\ &= i\sum_{m\in\mathbb{Z}} (\mathrm{sign} m)\hat{u}(m)e^{i\pi m x} \end{aligned} \tag{4.5}$$

if

$$u(x) \sim \sum_{m\in\mathbb{Z}} \hat{u}(m)e^{i\pi m x}. \tag{4.6}$$

It is clear from (4.5) that $\mathcal{H} : H^s \to H^s$ is isometric, i.e., $\|\mathcal{H}u\|_s = \|u\|_s$, that it maps even functions to odd functions and vice versa,

$$\mathcal{H} : H^s_e \to H^s_o, \qquad \mathcal{H} : H^s_o \to H^s_e, \tag{4.7}$$

and that

$$\mathcal{H}^2 = -I. \tag{4.8}$$

Now let $\mathcal{A}$ be the single-layer operator for an appropriately parametrised circle of radius $e^{-1/2}$,

$$\mathcal{A}u(x) = -\frac{1}{\pi}\int_{-1}^{1} \log|2e^{-1/2}\sin(\frac{\pi}{2}(x-y))|u(y)dy. \tag{4.9}$$

It is well known (see [1], [13]) that $\mathcal{A}$ is expressible as

$$\mathcal{A}u(x) = \frac{1}{\pi}\sum_{m\in\mathbb{Z}} \frac{\hat{u}(m)}{\max\{1,|m|\}}e^{i\pi m x} \tag{4.10}$$

if u has the Fourier representation (4.6), from which it is clear that

$$\mathcal{A} : H^s \to H^{s+1} \tag{4.11}$$

is an isometric operator, apart from an unimportant constant factor. From the Fourier representation it is also clear that

$$D\mathcal{A} = \mathcal{A}D = \mathcal{H},$$

where D is the operator of differentiation. From this we recover, on recalling (4.8) and (4.10),

$$\mathcal{A}^{-1} = -D\mathcal{H} + \mathcal{T} = -\mathcal{H}D + \mathcal{T}, \tag{4.12}$$

where $\mathcal{T} = \pi\int_0^1 u(y)dy$.

The Hilbert transform (4.5) can be written, using only properties of the trigonometric functions, as

$$\begin{aligned} \mathcal{H}u(x) &= \frac{1}{2}\int_{-1}^{1} \frac{\sin(\pi x)+\sin(\pi y)}{\cos(\pi x)-\cos(\pi y)}u(y)dy \\ &= \frac{1}{2}\int_{-1}^{1} \frac{\sin(\pi x)}{\cos(\pi x)-\cos(\pi y)}u(y)dy + \frac{1}{2}\int_{-1}^{1} \frac{\sin(\pi y)}{\cos(\pi x)-\cos(\pi y)}u(y)dy \\ &=: \mathcal{H}_e u(x) + \mathcal{H}_o u(x), \end{aligned} \tag{4.13}$$

where (because the kernel of $\mathcal{H}_e$ is even in y, and the kernel of $\mathcal{H}_o$ is odd in y), if $u = u_e + u_o$ with $u_e \in H_e^s$ and $u_o \in H_o^s$, then

$$\mathcal{H}_e u_o = 0, \quad \mathcal{H}_e u_e = \mathcal{H} u_e, \quad \mathcal{H}_o u_e = 0, \quad \mathcal{H}_o u_o = \mathcal{H} u_o. \tag{4.14}$$

Also important to us is the restriction of $\mathcal{A}$ to H_e^s. If $u_e \in H_e^s$ then because u_e is even we have, from (4.9),

$$\begin{aligned} \mathcal{A}u_e(x) &= -\frac{1}{2\pi}\int_{-1}^{1}\left(\log|2e^{-1/2}\sin(\frac{\pi}{2}(x-y))| + \log|2e^{-1/2}\sin(\frac{\pi}{2}(x+y))|\right)u_e(y)dy \\ &= -\frac{1}{\pi}\int_0^1 \log|2e^{-1}(\cos(\pi x) - \cos(\pi y)|u_e(y)dy \\ &=: \mathcal{A}_e u_e(x). \end{aligned} \tag{4.15}$$

From (4.10) we then have

$$\mathcal{A}_e u(x) = \frac{2}{\pi}\sum_{m\in\mathbb{Z}^+}{}' \frac{\hat{u}(m)}{\max\{1,m\}}\cos(\pi m x), \tag{4.16}$$

where, from (4.6),

$$u_e(x) \sim 2\sum_{m\in\mathbb{Z}^+}{}' \hat{u}_e(m)\cos(\pi m x), \tag{4.17}$$

and $\mathbb{Z}^+ = \{0,1,2,\ldots\}$, and the prime indicates that the $m=0$ term is to be multiplied by $1/2$.

Finally, we see from (4.12), (4.14), (4.15) that, as an operator on H_e^s,

$$\mathcal{A}_e^{-1} = -D\mathcal{H}_e + \mathcal{T} = -\mathcal{H}_o D + \mathcal{T}. \tag{4.18}$$

The last relation will play an important role in the subsequent analysis.

4.2 The collocation projection

Let us define a space of 2-periodic cosine functions of degree $N-1$,

$$T_{e,h} = \operatorname{span}\{\cos(\pi m x) : 0 \le m \le N-1\}. \tag{4.19}$$

From here on, we set $\psi_m(x) := \cos(\pi m x)$.

We introduce a collocation projection P_h from H_e^s (with $s > 1/2$) to $T_{e,h}$, that is similar but not identical to the one introduced in [1]:

$$P_h f = 2\sum_{k=0}^{N-1}{}'(f,\psi_k)_h\psi_k, \tag{4.20}$$

where

$$(f,g)_h = h\sum_{k=0}^{N-1}(f\cdot\overline{g})(kh + h/2).$$

In the next lemma, we introduce several interesting properties of P_h. It turns out that P_h is an interpolatory projection operator.

Lemma 4.1 *The operator P_h satisfies the following properties as an operator on H_e^s, $s > 1/2$. Let $f \in H_e^s$ with $s > 1/2$. Then*

(1) $(P_h f, \psi)_h = (f, \psi)_h, \quad \psi \in T_{e,h}$.

(2) $P_h^2 = P_h$.

(3) $P_h f(kh + h/2) = f(kh + h/2), \quad k = 0, \cdots, N-1,\ h = 1/N$.

(4)

$$\|f - P_h f\|_t \le C h^{s-t} \|f\|_s \ \text{ for } s > 1/2, s \ge t \ge 0. \tag{4.21}$$

Proof. The property (1) follows from the definition of $P_h f$ and the easily verified 'discrete-orthogonality' property

$$(\psi_k, \psi_j)_h = a_k \delta_{kj} \text{ for } 0 \le k, j \le N-1, \tag{4.22}$$

with $a_0 = 1$ and $a_k = 1/2$ for $1 \le k \le N-1$. Property (2) follows from property (1). To prove (3) it is useful to define first the $N \times N$ matrix M with elements

$$m_{kl} = \begin{cases} \sqrt{h} & \text{if } k = 0, \\ \sqrt{2h}\,\psi_k(lh + h/2) & \text{if } 1 \le k \le N-1, \end{cases}$$

and $0 \le l \le N-1$. Then (4.22) is equivalent to $MM^* = I$, from which it follows that $M^* M = I$, or

$$2 \sum_{k=0}^{N-1}{}' \psi_k(lh + h/2) \bar{\psi}_k(l'h + h/2) = \delta_{ll'}. \tag{4.23}$$

(This identity can of course also be established directly.) The property (3) follows immediately from (4.23). The approximation property (4) is standard, see [1]. □

Remark 1 *The projection in [1] is also a collocation projection at evenly spaced node points, but in that work the nodes are not located symmetrically on* $[0, 1]$*, because whereas* 0 *is a node,* 1 *is a 'midpoint'. Here our collocation is a simple midpoint collocation, and the nodes are symmetrically located.*

With the help of the projection P_h, the collocation method of this paper can be expressed as: find $z_1^h, z_2^h \in T_{e,h}$ such that

$$P_h(\mathcal{V}_{11} z_1^h + \mathcal{V}_{12} z_2^h) = P_h f, \tag{4.24}$$

$$P_h \mathcal{K}_{21} z_1^h + z_2^h + P_h \mathcal{K}_{22} z_2^h = P_h g. \tag{4.25}$$

4.3 Mellin convolution operators

We recall some results on Mellin convolution operators defined on the half axis or on the unit interval. These are based on [3], [4], [5] and [7].

(i) The Mellin transform $\widehat{v}$ of a function $v : \mathbb{R}^+ \to \mathbb{C}$ is defined as

$$\widehat{v}(z) = \int_0^\infty s^{iz-1} v(s) ds.$$

The operator $v \to \widehat{v}$ is an isometric isomorphism of $L_2(\mathbb{R}^+)$ onto $L_2(\{\mathrm{Im}\ z = -1/2\})$, and its inverse is

$$v(s) = \frac{1}{2\pi} \int_{\mathrm{Im}\ z=-1/2} s^{-iz} \widehat{v}(z) |dz|.$$

(ii) If $\mathcal{K}$ is a Mellin convolution operator, i.e

$$\mathcal{K}v(t) = \int_0^\infty K(\frac{t}{s}) \frac{v(s)}{s} ds \tag{4.26}$$

with kernel $s^{-1/2}K(s) \in L_1(\mathbb{R}^+)$, then $\widehat{\mathcal{K}v}(z) = \widehat{K}(z) \cdot \widehat{v}(z)$, and $\mathcal{K}$ is a continuous operator on $L_2(\mathbb{R}^+)$ with norm bounded by

$$\|\mathcal{K}\|_0 \leq \sup_{\mathrm{Im}\ z=-1/2} |\widehat{K}(z)|. \tag{4.27}$$

Note that this extends to more general operators of the form (4.26) provided the Mellin transform is bounded on $\mathrm{Im}\ z = -1/2$; cf. e.g. the operators $\widetilde{\mathcal{H}}_o$ and $\widetilde{\mathcal{H}}_e$ defined below in (5.4).

From here on, we abuse notation by defining:

$$\widehat{\mathcal{K}}(z) := \mathrm{symbol}(\mathcal{K}) = \widehat{K}(z).$$

If $\mathcal{K}$ and $\mathcal{L}$ are Mellin convolution operators with bounded symbols on $\mathrm{Im}\ z = -1/2$, then $\widehat{\mathcal{K}\mathcal{L}}(z) = \widehat{\mathcal{K}}(z) \cdot \widehat{\mathcal{L}}(z)$.

(iii) The symbol $\widehat{\mathcal{K}}(z)$ of the Mellin convolution operator (4.26) is said to be of class $\Sigma^{-\infty}_{\alpha,\beta}$, $\alpha < -1/2 < \beta$, if it is analytic in the strip $\alpha < \mathrm{Im}\ z < \beta$ and if the estimates

$$\widehat{\mathcal{K}}(z) = O((1+|z|)^{-k}),\ |z| \to \infty,\ k \in \mathbb{Z}^+$$

hold uniformly in each substrip $\alpha' < \mathrm{Im}\ z < \beta'$, $\alpha < \alpha' < -1/2 < \beta' < \beta$. Then the kernel function $K(s)$ of $\mathcal{K}$ satisfies the estimates

$$\sup_{s \in \mathbb{R}^+} |s^{k-\rho} D^k K(s)| < \infty,\ k \in \mathbb{Z}^+,\ \alpha < \rho < \beta. \tag{4.28}$$

In particular, (4.28) implies $s^{-1/2}K(s) \in L_1(\mathbb{R}^+)$ so that $\mathcal{K}$ is a bounded operator on $L_2(\mathbb{R}^+)$ satisfying the estimate (4.27).

(iv) Let χ be a smooth function with $\mathrm{supp}(\chi) \subset [0,1]$, and let ψ be a bounded function such that $\mathrm{supp}(\psi) \subset [0,1]$ and $\psi(s) = 0, s \in [0,\epsilon]$, for some $\epsilon \in (0,1)$. If $\widehat{\mathcal{K}} \in \Sigma^{-\infty}_{\alpha,\beta}$ for some $\alpha < -1/2 < \beta$ and $\mathcal{K}$ is the corresponding Mellin convolution operator (4.26), then the operators $\chi\mathcal{K} - \mathcal{K}\chi I$ and $\psi\mathcal{K}$ are Hilbert–Schmidt and hence compact on $L_2(\mathbb{R}^+)$.

We finally recall standard results on the invertibility of a convolution operator $I + \mathcal{K}$ restricted to the unit interval and on the stability of a corresponding finite section method. Note that, with the isometry $\mathcal{J} : L_2(0,1) \to L_2(\mathbb{R}^+)$ defined by $(\mathcal{J}v)(t) = v(e^{-t})e^{-t/2}$, $\mathcal{J}\mathcal{K}\mathcal{J}^{-1}$ is a Wiener–Hopf integral operator with kernel function $e^{-t/2}K(e^{-t}) \in L_1(\mathbb{R})$. Thus the following assertions are easily checked via known results on Wiener–Hopf operators ([8]).

(v) Let ϕ and ϕ_r, $0 < r < 1$, be the characteristic functions of the intervals $(0,1)$ and $(r,1)$, respectively. Suppose the conditions $s^{-1/2}K(s) \in L_1(\mathbb{R}^+)$ and

$$1 + \widehat{\mathcal{K}}(-i/2 + y) \neq 0,\ y \in \mathbb{R};\ \{\arg(1 + \widehat{\mathcal{K}}(-i/2 + y))\}_{-\infty}^{\infty} = 0$$

are satisfied, where $\{\arg\cdot\}_{-\infty}^{\infty}$ denotes the variation of the argument when y runs from $-\infty$ to ∞. Then the Mellin convolution operator $\phi(I + \mathcal{K})\phi$ is continuously invertible on $L_2(0,1)$ and the corresponding finite section operators $\phi_r(I + \mathcal{K})\phi_r$ are stable, i.e., there is an $r_0 > 0$ and a $c > 0$ such that

$$\|\phi_r(I + \mathcal{K})\phi_r v\|_0 \geq c\|\phi_r v\|_0,\ v \in L_2(0,1),$$

for any $r \leq r_0$.

5 Mapping properties of integral operators and Mellin techniques

Write (3.13) in the form:

$$\mathbf{B}\mathbf{z} = (\mathbf{A} + \mathbf{K})\,\mathbf{z} = \mathbf{f}, \tag{5.1}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathcal{A}_e & \mathcal{V}_{12} \\ 0 & I \end{bmatrix},\quad \mathbf{K} = \begin{bmatrix} \mathcal{V}_{11} - \mathcal{A}_e & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix}, \tag{5.2}$$

and

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},\quad \mathbf{f} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{5.3}$$

Note that $\mathcal{K}_{22}$ is a compact operator mapping even functions to even functions since it has a continuous kernel.

In this section we analyse the operators in (5.2) by use of localization and Mellin transformation techniques.

The key to the analysis that follows is the recognition that the difficulties with the integral equation (5.1) (which in explicit form is (3.7), (3.8)) arise only when x and y are both near 0, or both near 1, i.e. the values of the parameter that correspond to junctions between the arcs Γ_D and Γ_N. In such a neighbourhood the kernels of each operator behave like a Mellin convolution. Therefore cut-off functions are introduced, which allow the operator to be separated into Mellin convolutions, describing all corner effects, and smooth remainders.

Let us introduce smooth cut-off functions χ_0, χ_1 on $[0,1]$ and ν_0, ν_1 on $[1,2]$, such that for some $0<\epsilon<1/2$,

$$\begin{array}{ll} \chi_0(x)=1,\ x\in[0,\epsilon],\ \ \mathrm{supp}(\chi_0)\subset[0,1/2), & \chi_1(x)=1,\ x\in[1-\epsilon,1],\ \ \mathrm{supp}(\chi_1)\subset(1/2,1], \\ \nu_1(x)=1,\ x\in[1,1+\epsilon],\ \ \mathrm{supp}(\nu_1)\subset[1,3/2), & \nu_0(x)=1,\ x\in[2-\epsilon,2],\ \ \mathrm{supp}(\nu_0)\subset(3/2,2]. \end{array}$$

Each of χ_0, χ_1, ν_0 and ν_1 is extended to a 2-periodic even function by expressions analogous to (3.4)–(3.6).

We also introduce certain Mellin convolution operators on the half axis $(0,\infty)$. (For further discussion of the Mellin transform and Mellin convolution operators, see [3], [5], [6].) Let us define

$$\begin{aligned} \widetilde{\mathcal{H}}_o u(x) &= \frac{1}{\pi}\int_0^\infty H_o(\frac{x}{y})\frac{u(y)}{y}dy, \\ \widetilde{\mathcal{H}}_e u(y) &= \frac{1}{\pi}\int_0^\infty H_e(\frac{x}{y})\frac{u(y)}{y}dy, \\ \widetilde{\mathcal{L}}_\omega u(x) &= \frac{1}{\pi}\int_0^\infty L_\omega(\frac{x}{y})\frac{u(y)}{y}dy, \\ \widetilde{\mathcal{K}}_\omega u(x) &= \frac{1}{\pi}\int_0^\infty K_\omega(\frac{x}{y})u\frac{u(y)}{y}dy, \end{aligned} \tag{5.4}$$

where

$$H_o(t)=\frac{2}{1-t^2},\qquad H_e(t)=\frac{2t}{1-t^2},$$
$$L_\omega(t)=-\frac{qt^{q-1}(t^q-\cos(\omega))}{t^{2q}-2t^q\cos(\omega)+1},\qquad K_\omega(t)=-\frac{qt^{q-1}\sin(\omega)}{t^{2q}-2t^q\cos(\omega)+1}.$$

It is worth noting that if $q=2$ then $\widetilde{\mathcal{L}}_0=\widetilde{\mathcal{H}}_e$.

It is convenient to extend the kernels of these operators to the whole real line in the following way: H_e and L_ω are extended to be odd functions, and H_o and K_ω are extended to even functions. It is then clear that $\widetilde{\mathcal{H}}_e u$ and $\widetilde{\mathcal{L}}_\omega u$ are odd, while $\widetilde{\mathcal{H}}_o u$ and $\widetilde{\mathcal{K}}_\omega u$ are even.

The symbols of these Mellin operators are (see [3], [6])

$$\begin{aligned} \widehat{\widetilde{\mathcal{H}}}_o(z) &= i\coth(\pi\frac{z}{2}), \\ \widehat{\widetilde{\mathcal{H}}}_e(z) &= i\coth(\pi\frac{z+i}{2}), \\ \widehat{\widetilde{\mathcal{L}}}_\omega(z) &= i\frac{\cosh((\pi-\omega)(z+i)/q)}{\sinh(\pi(z+i)/q)}, \\ \widehat{\widetilde{\mathcal{K}}}_\omega(z) &= \frac{\sinh((\pi-\omega)(z+i)/q)}{\sinh(\pi(z+i)/q)}. \end{aligned} \tag{5.5}$$

The integral operators in (5.2) can now be expressed as in the following lemma. In this lemma, and throughout the paper, $\mathcal{E}$ denotes a generic compact operator, which may be different in its different appearances. In the first term of the first result, property (1), it is understood that the domains of the Mellin operators $\widetilde{\mathcal{L}}_0$ and $\widetilde{\mathcal{H}}_e$ are restricted to a finite interval in the natural way. In the second term of property (1) the double tilde on $\widetilde{\widetilde{\mathcal{L}}}_0$ indicates that the transformations $x \mapsto 1-x$ and $y \mapsto 1-y$ are to be carried out, corresponding to the fact that in this term the singularity is not at $x=0$ and $y=0$ but at $x=1$ and $y=1$. The double-tilde notation in the remaining terms is to be understood in an analogous way, with the precise transformations in each case being apparent from the proofs.

Lemma 5.1 *As operators on even functions,*

(1)

$$D(\mathcal{V}_{11}-\mathcal{A}_e)=\chi_0(\widetilde{\mathcal{L}}_0-\widetilde{\mathcal{H}}_e)\chi_0-\chi_1(\widetilde{\widetilde{\mathcal{L}}}_0-\widetilde{\widetilde{\mathcal{H}}}_e)\chi_1+\mathcal{E},$$

(2)

$$D\mathcal{V}_{12}=\chi_0\widetilde{\widetilde{\mathcal{L}}}_{\omega_0}\nu_0-\chi_1\widetilde{\widetilde{\mathcal{L}}}_{\omega_1}\nu_1+\mathcal{E},$$

(3)

$$\mathcal{K}_{21}=\nu_0\widetilde{\widetilde{\mathcal{K}}}_{\omega_0}\chi_0+\nu_1\widetilde{\widetilde{\mathcal{K}}}_{\omega_1}\chi_1+\mathcal{E}.$$

And as an operator on odd functions,

(4)

$$\mathcal{H}_o=\chi_0\widetilde{\mathcal{H}}_o\psi_0-\chi_1\widetilde{\widetilde{\mathcal{H}}}_o\psi_1+\mathcal{H}_o(1-\chi_0-\chi_1)+\mathcal{E},$$

where ψ_0 and ψ_1 are suitable cut-off functions such that $\psi_0\chi_0=\chi_0$, $\psi_1\chi_1=\chi_1$, $\psi_0\chi_1=0$ and $\psi_1\chi_0=0$.

Proof. The results all follow from the asymptotic behaviour of the kernel of the integral operators. First, by the definition of α, α_1 and α_2 in (2.1), (2.2) and (3.1)–(3.3) we can assume that

$$\begin{cases} \alpha_1(x)-\alpha_1(0)=C_0x^q, & 0\le x\le\epsilon, \\ \alpha_2(x)-\alpha_2(0)=C_0e^{i\omega_0}(2-x)^q, & 2-\epsilon\le x\le 2, \\ \alpha_1(x)-\alpha_1(1)=C_1(1-x)^q, & 1-\epsilon\le x\le 1, \\ \alpha_2(x)-\alpha_2(1)=C_1e^{-i\omega_1}(x-1)^q, & 1\le x\le 1+\epsilon, \end{cases} \tag{5.6}$$

where ω_0 and ω_1 are the interior angles at the corners corresponding to $x=0$ and $x=1$ respectively and C_0 and C_1 are complex constants. (Points in $\mathbb{R}^2$ are here identified with complex numbers in the usual way.) Then for $\phi\in H_e^0$ we have, from (3.9) and (4.15),

$$D(\mathcal{V}_{11}-\mathcal{A}_e)\phi(x)=-\frac{1}{\pi}\int_0^1\mathrm{Re}\left(\frac{(\alpha_1(x)-\alpha_1(y),\alpha_1'(x))}{|\alpha_1(x)-\alpha_1(y)|^2}+\frac{\pi\sin(\pi x)}{\cos(\pi x)-\cos(\pi y)}\right)\phi(y)dy.$$

Noting that the apparent singularities at $x = y$ in the two terms of the kernel cancel, we see that

$$\begin{aligned}
D(\mathcal{V}_{11} - \mathcal{A}_e)\phi(x) &= -\frac{1}{\pi}\int_0^{\epsilon} \chi_0(x)\chi_0(y)\left(\frac{qx^{q-1}}{x^q - y^q} - \frac{2x}{x^2 - y^2}\right)\phi(y)dy \\
&\quad + \frac{1}{\pi}\int_{1-\epsilon}^{1} \chi_1(x)\chi_1(y)\left(\frac{q(1-x)^{q-1}}{(1-x)^q - (1-y)^q} - \frac{2(1-x)}{(1-x)^2 - (1-y)^2}\right)\phi(y)dy \\
&\quad + \text{smoother terms} \\
&= -\frac{1}{\pi}\int_0^{\infty} \chi_0(x)\chi_0(y)\left(\frac{qx^{q-1}}{x^q - y^q} - \frac{2x}{x^2 - y^2}\right)\phi(y)dy \\
&\quad + \frac{1}{\pi}\int_0^{\infty} \chi_1(x)\chi_1(y)\left(\frac{q\tilde{x}^{q-1}}{\tilde{x}^q - \tilde{y}^q} - \frac{2\tilde{x}}{\tilde{x}^2 - \tilde{y}^2}\right)\phi(1-\tilde{y})d\tilde{y}\Big|_{\{\tilde{x}=1-x,\ \tilde{y}=1-y\}} \\
&\quad + \text{ smoother terms.}
\end{aligned}$$

Then (1) follows.

By the same argument,

$$\begin{aligned}
D\mathcal{V}_{12}\phi(x) &= -\frac{1}{\pi}\int_1^2 \mathrm{Re}\left(\frac{(\alpha_1(x) - \alpha_2(y), \alpha_1'(x))}{|\alpha_1(x) - \alpha_2(y)|^2}\right)\phi(y)dy \\
&= \frac{1}{\pi}\int_1^{1+\epsilon} \chi_1(x)\nu_1(y)\frac{q(1-x)^{q-1}((1-x)^q - (y-1)^q\cos(\omega_1))}{(1-x)^{2q} - 2(1-x)^q(y-1)^q\cos(\omega_1) + (y-1)^{2q}}\phi(y)dy \\
&\quad - \frac{1}{\pi}\int_{2-\epsilon}^{2} \chi_0(x)\nu_0(y)\frac{qx^{q-1}(x^q - (2-y)^q\cos(\omega_0))}{x^{2q} - 2x^q(2-y)^q\cos(\omega_0) + (2-y)^{2q}}\phi(y)dy \\
&\quad + \text{smoother terms} \\
&= \frac{1}{\pi}\int_0^{\infty} \chi_1(x)\nu_1(y)\frac{q\tilde{x}^{q-1}(\tilde{x}^q - \tilde{y}^q\cos(\omega_1))}{\tilde{x}^{2q} - 2\tilde{x}^q\tilde{y}^q\cos(\omega_1) + \tilde{y}^{2q}}\phi(\tilde{y}+1)d\tilde{y}\Big|_{\{\tilde{x}=1-x,\ \tilde{y}=y-1\}} \\
&\quad - \frac{1}{\pi}\int_0^{\infty} \chi_0(x)\nu_0(y)\frac{q\tilde{x}^{q-1}(\tilde{x}^q - \tilde{y}^q\cos(\omega_0))}{\tilde{x}^{2q} - 2\tilde{x}^q\tilde{y}^q\cos(\omega_0) + \tilde{y}^{2q}}\phi(2-\tilde{y})d\tilde{y}\Big|_{\{\tilde{x}=x,\ \tilde{y}=2-y\}} \\
&\quad + \text{smoother terms,}
\end{aligned}$$

which proves (2). Similarly, to prove (3),

$$\begin{aligned}
\mathcal{K}_{21}\phi(x) &= -\frac{1}{\pi}\int_0^1 \mathrm{Re}\left(\frac{|\alpha_2'(x)|(\alpha_2(x) - \alpha_1(y), n_x)}{|\alpha_2(x) - \alpha_1(y)|^2}\right)\phi(y)dy, \\
&= -\frac{1}{\pi}\int_0^{\epsilon} \nu_0(x)\chi_0(y)\frac{q(2-x)^{q-1}y^q\sin(\omega_0)}{(2-x)^{2q} - 2(2-x)^q y^q\cos(\omega_0) + y^{2q}}\phi(y)dy \\
&\quad - \frac{1}{\pi}\int_{1-\epsilon}^{1} \nu_1(x)\chi_1(y)\frac{q(x-1)^{q-1}(1-y)^q\sin(\omega_1)}{(x-1)^{2q} - 2(x-1)^q(1-y)^q\cos(\omega_1) + (1-y)^{2q}}\phi(y)dy \\
&\quad + \text{smoother terms} \\
&= -\frac{1}{\pi}\int_0^{\infty} \nu_0(x)\chi_0(y)\frac{q\tilde{x}^{q-1}\tilde{y}^q\sin(\omega_0)}{\tilde{x}^{2q} - 2\tilde{x}^q\tilde{y}^q\cos(\omega_0) + \tilde{y}^{2q}}\phi(\tilde{y})d\tilde{y}\Big|_{\{\tilde{x}=2-x,\ \tilde{y}=y\}}
\end{aligned}$$

$$
- \frac{1}{\pi} \int_0^\infty \nu_1(x)\chi_1(y) \frac{q\tilde{x}^{q-1}\tilde{y}^q \sin(\omega_1)}{\tilde{x}^{2q} - 2\tilde{x}^q\tilde{y}^q\cos(\omega_1) + \tilde{y}^{2q}} \phi(1-\tilde{y})d\tilde{y}\Bigg|_{\{\tilde{x}=x-1,\ \tilde{y}=1-y\}}
$$
$$
+ \text{smoother terms.}
$$

The proof of (4) follows in the same way as above, using the fact that the commutator of χI and $\mathcal{H}_o$ is an integral operator with smooth kernel for any smooth 2-periodic even function χ. □

Remark 2 *It is easily seen from (5.5) that the symbols of the Mellin convolution operators* $\widetilde{\mathcal{L}}_0 - \widetilde{\mathcal{H}}_e$ *and* $\widetilde{\mathcal{L}}_\omega$, $\widetilde{\mathcal{K}}_\omega$, $0 < \omega < 2\pi$, *are of class* $\Sigma^{-\infty}_{-1,0}$. *For* $q \geq 2$, *these symbols even belong to* $\Sigma^{-\infty}_{-1,1}$.

Lemma 5.2 *The operator* $\mathbf{A}$ *defined in (5.2) is an isomorphism of* $H^0_e \times H^0_e$ *onto* $H^1_e \times H^0_e$, *with inverse given by*

$$
\mathbf{A}^{-1} = \left[\begin{array}{cc} \mathcal{A}_e^{-1} & -\mathcal{A}_e^{-1}\mathcal{V}_{12} \\ 0 & I \end{array} \right]. \tag{5.7}
$$

Proof. By Lemma 5.1 (2), Remark 2 and §4.3(iii), $\mathcal{V}_{12} : H^0_e \to H^1_e$ is bounded. Hence (5.7) is a bounded operator of $H^1_e \times H^0_e$ into $H^0_e \times H^0_e$, which is easily seen to be the inverse of $\mathbf{A}$. □

To investigate the solvability of Equation (5.1), we consider the operator

$$
\mathbf{A}^{-1}\mathbf{B} = \mathbf{I} + \mathbf{M},\ \mathbf{M} := \mathbf{A}^{-1}\mathbf{K} = \left[\begin{array}{cc} \mathcal{M} & \mathcal{E} \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{array} \right], \tag{5.8}
$$

where

$$
\mathcal{M} =: \mathcal{A}_e^{-1}(\mathcal{V}_{11} - \mathcal{A}_e) - \mathcal{A}_e^{-1}\mathcal{V}_{12}\mathcal{K}_{21},\ \mathcal{E} := -\mathcal{A}_e^{-1}\mathcal{V}_{12}\mathcal{K}_{22}. \tag{5.9}
$$

Note that $\mathcal{M}$ and $\mathcal{K}_{21}$ are bounded operators on H^0_e while $\mathcal{E}$ and $\mathcal{K}_{22}$ are compact; see Lemma 5.1, Remark 2 and §4.3(iii). With the notation of Lemma 5.1 (except that we now put aside the double tilde notation), we have:

Lemma 5.3 *As an operator on even functions,*

$$
\mathcal{M} = \chi_0\widetilde{\mathcal{M}}_0\chi_0 + \chi_1\widetilde{\mathcal{M}}_1\chi_1 + \mathcal{E}, \tag{5.10}
$$

where the symbols of the Mellin convolution operators $\widetilde{\mathcal{M}}_j$, $j = 0, 1$, *take the form*

$$
\widehat{\widetilde{\mathcal{M}}}_j(z) = -\widehat{\widetilde{\mathcal{H}}}_o(z)[\widehat{\widetilde{\mathcal{L}}}_0(z) - \widehat{\widetilde{\mathcal{H}}}_e(z)] + \widehat{\widetilde{\mathcal{H}}}_o(z)\widehat{\widetilde{\mathcal{L}}}_{\omega_j}(z)\widehat{\widetilde{\mathcal{K}}}_{\omega_j}(z) \tag{5.11}
$$

and are of class $\Sigma^{-\infty}_{-1,0}$.

Proof. From (4.18) and Lemma 5.1, (1) and (4), we obtain

$$\begin{aligned}
\mathcal{A}_e^{-1}(\mathcal{V}_{11}-\mathcal{A}_e) &= -\mathcal{H}_o D(\mathcal{V}_{11}-\mathcal{A}_e)+\mathcal{E} \\
&= -\mathcal{H}_o[\chi_0(\widetilde{\mathcal{L}}_0-\widetilde{\mathcal{H}}_e)\chi_0-\chi_1(\widetilde{\mathcal{L}}_0-\widetilde{\mathcal{H}}_e)\chi_1+\mathcal{E}]+\mathcal{E} \\
&= -\chi_0\widetilde{\mathcal{H}}_o(\widetilde{\mathcal{L}}_0-\widetilde{\mathcal{H}}_e)\chi_0-\chi_1\widetilde{\mathcal{H}}_o(\widetilde{\mathcal{L}}_0-\widetilde{\mathcal{H}}_e)\chi_1+\mathcal{E},
\end{aligned}$$

where we have used the compactness results of §4.3(iv). Analogously,

$$\begin{aligned}
\mathcal{A}_e^{-1}\mathcal{V}_{12}\mathcal{K}_{21} &= -\mathcal{H}_o D\mathcal{V}_{12}\mathcal{K}_{21}+\mathcal{E} \\
&= -\mathcal{H}_o[\chi_0\widetilde{\mathcal{L}}_{\omega_0}\nu_0-\chi_1\widetilde{\mathcal{L}}_{\omega_1}\nu_1+\mathcal{E}][\nu_0\widetilde{\mathcal{K}}_{\omega_0}\chi_0+\nu_1\widetilde{\mathcal{K}}_{\omega_1}\chi_1+\mathcal{E}]+\mathcal{E} \\
&= -\chi_0\widetilde{\mathcal{H}}_o\widetilde{\mathcal{L}}_{\omega_0}\widetilde{\mathcal{K}}_{\omega_0}\chi_0-\chi_1\widetilde{\mathcal{H}}_o\widetilde{\mathcal{L}}_{\omega_1}\widetilde{\mathcal{K}}_{\omega_1}\chi_1+\mathcal{E},
\end{aligned}$$

where we have used (4.18), Lemma 5.1, (2), (3) and (4), and the compactness results of §4.3(iv).

Combining the above relations with (5.4), (5.5) and (5.9), we get (5.10) and (5.11). The last assertion of the lemma follows from Remark 2 and the fact that $\widehat{\widetilde{\mathcal{H}}_o}(z)$ is analytic and (together with all its derivatives) bounded on each strip $-1+\delta<\mathrm{Im}\ z<-\delta$, $\delta\in(0,1/2)$. □

Lemma 5.4 *For $q\geq 2$, $I+\mathcal{M}$ is a Fredholm operator of index 0 on H_e^0.*

Proof. Let ϕ_0 and ϕ_1 denote the characteristic functions of the intervals (0,1/2) and (1/2,1), respectively, extended to 2-periodic even functions. From (5.10) and §4.3(iv), we obtain the representation

$$I+\mathcal{M}=\phi_0(I+\widetilde{\mathcal{M}}_0)\phi_0+\phi_1(I+\widetilde{\mathcal{M}}_1)\phi_1+\mathcal{E}. \tag{5.12}$$

To prove the assertion, it is obviously sufficient to verify the invertibility of the Mellin convolution operators $\phi_0(I+\widetilde{\mathcal{M}}_0)\phi_0$ and $\phi_1(I+\widetilde{\mathcal{M}}_1)\phi_1$ on $L_2(0,1/2)$ and $L_2(1/2,1)$, respectively, and we shall do this for the first term without loss of generality. In view of §4.3(v) we have to show that

$$\{\arg(1+\widehat{\widetilde{\mathcal{M}}_0}(y-i/2))\}_{-\infty}^{\infty}=0. \tag{5.13}$$

From (5.11) and the identity $\widehat{\widetilde{\mathcal{H}}_o}\widehat{\widetilde{\mathcal{H}}_e}=-1$, we have

$$1+\widehat{\widetilde{\mathcal{M}}_0}(z)=-\widehat{\widetilde{\mathcal{H}}_o}(z)\widehat{\widetilde{\mathcal{L}}_0}(z)[1-\widehat{\widetilde{\mathcal{L}}_{\omega_0}}(z)\widehat{\widetilde{\mathcal{L}}_0}(z)^{-1}\widehat{\widetilde{\mathcal{K}}_{\omega_0}}(z)].$$

To check (5.13), it is now enough to prove the estimates

$$\mathrm{Re}\{-\widehat{\widetilde{\mathcal{H}}_o}(z)\widehat{\widetilde{\mathcal{L}}_0}(z)\}\geq c>0,\ \mathrm{Im}\ z=-1/2, \tag{5.14}$$

$$|\widehat{\widetilde{\mathcal{L}}_{\omega_0}}(z)\widehat{\widetilde{\mathcal{L}}_0}(z)^{-1}\widehat{\widetilde{\mathcal{K}}_{\omega_0}}(z)|\leq C<1,\ \mathrm{Im}\ z=-1/2. \tag{5.15}$$

By a simple calculation,

$$\begin{aligned}
-(\widehat{\widetilde{\mathcal{H}}_o}\widehat{\widetilde{\mathcal{L}}_0})(y-i/2) &= \frac{\sinh(\pi y)+i}{\cosh(\pi y)}\cdot\frac{\sinh(2\pi y/q)-i\sin(\pi/q)}{\cosh(2\pi y/q)-\cos(\pi/q)}, \\
\mathrm{Re}\{-(\widehat{\widetilde{\mathcal{H}}_o}\widehat{\widetilde{\mathcal{L}}_0})(y-i/2)\} &= \frac{\sinh(\pi y)\sinh(2\pi y/q)+\sin(\pi/q)}{\cosh(\pi y)(\cosh(2\pi y/q)-\cos(\pi/q))},
\end{aligned}$$

which implies (5.14) for any $q > 1$. To prove (5.15), we observe that

$$\widehat{\mathcal{L}}_\omega(z)\widehat{\mathcal{L}}_0(z)^{-1}\widehat{\mathcal{K}}_\omega(z) = \frac{\cosh((\pi-\omega)(z+i)/q)}{\cosh(\pi(z+i)/q)} \cdot \frac{\sinh((\pi-\omega)(z+i)/q)}{\sinh((\pi(z+i)/q)} = a(2(z+i)/q),$$

where $a(z) := \sinh((\pi-\omega)z)/\sinh(\pi z)$ is the symbol of the double layer potential in case of the arc-length parametrisation, which satisfies (see [3], [2])

$$\sup_{y\in\mathbb{R}} |a(i\gamma + y)| < 1 \text{ for } |\gamma| \leq 1/2.$$

Thus we obtain the desired result whenever $q \geq 2$. □

Corollary 5.5 *Assume (A1) and $q \geq 2$. Then the operator $\mathbf{B} : H_e^0 \times H_e^0 \to H_e^1 \times H_e^0$ has a bounded inverse.*

Proof. First we observe that the operator

$$\mathbf{I}+\mathbf{M} = \begin{bmatrix} I+\mathcal{M} & \mathcal{E} \\ \mathcal{K}_{21} & I+\mathcal{K}_{22} \end{bmatrix} : H_e^0 \times H_e^0 \to H_e^0 \times H_e^0$$

is Fredholm with index 0, using Lemma 5.4 and the compactness of $\mathcal{E}$ and $\mathcal{K}_{22}$. Thus, by Lemma 5.2, $\mathbf{B}$ is a Fredholm operator with index 0. So it suffices to show that $\mathbf{Bz} = \mathbf{0}$ and $\mathbf{z} \in H_e^0 \times H_e^0$ imply $\mathbf{z} = \mathbf{0}$. We now proceed as in the proof of Theorem 2 in [6] and consider the function

$$Z(P) := |(\alpha^{-1})'(P)|\mathbf{z}(\alpha^{-1})(P)),\ P \in \Gamma,$$

where $\alpha^{-1} : \Gamma \to [0,2]$ is the inverse transformation of (2.2). Then Z solves the homogeneous version of the original integral equations (1.4) and satisfies (cf. [6]) $Z \in L_p(\Gamma)$ for some $p > 1$ sufficiently close to 1. Hence $Z = 0$ by (A1), which implies $\mathbf{z} = \mathbf{0}$. □

Finally, for the convergence analysis of §6, we need a stability result for a finite section method applied to the operator $\mathbf{I}+\mathbf{M}$ defined in (5.8). Introduce, for $v \in H_e^0$ and $0 < r < 1/2$, the truncation $T_r v$ as the 2-periodic even extension of

$$T_r v(x) = \begin{cases} v(x), & x \in (r, 1-r) \\ 0, & x \in (0,r) \cup (1-r, 1). \end{cases} \tag{5.16}$$

The finite section approximation to $\mathbf{M}$ is then defined to be

$$\mathbf{M}_r = \begin{bmatrix} \mathcal{M}T_r & \mathcal{E} \\ \mathcal{K}_{21}T_r & \mathcal{K}_{22} \end{bmatrix}.$$

Lemma 5.6 *There exists $r_0 > 0$ such that*

$$\|(\mathbf{I}+\mathbf{M}_r)\mathbf{v}\|_{H_e^0\times H_e^0} \geq c\|\mathbf{v}\|_{H_e^0\times H_e^0},\ \mathbf{v} \in H_e^0 \times H_e^0,$$

for any $r \leq r_0$.

Proof. Using (5.12), we may write $\mathbf{M}_r = \mathbf{N}_r + \mathcal{F}_r$, where

$$\begin{aligned} \mathbf{N}_r &= \begin{bmatrix} \mathcal{N}T_r & 0 \\ \mathcal{K}_{21}T_r & 0 \end{bmatrix}, \ \mathcal{F}_r = \begin{bmatrix} \mathcal{E}T_r & \mathcal{E} \\ 0 & \mathcal{K}_{22} \end{bmatrix}, \\ \mathcal{N} &= \phi_0\widetilde{\mathcal{M}}_0\phi_0 + \phi_1\widetilde{\mathcal{M}}_1\phi_1. \end{aligned}$$

Note that $I+\mathcal{N}$ is invertible on H_e^0 (cf. the proof of Lemma 5.4) while $\mathbf{I}+\mathbf{M}$ is invertible on $H_e^0 \times H_e^0$ by Corollary 5.5 and Lemma 5.2. Further, since T_r converges strongly to the identity as $r \to 0$ and the operators $\mathcal{E}$ and $\mathcal{K}_{22}$ are compact, a standard perturbation result (cf. [12], Chap. 17.1) reduces the assertion of the lemma to the corresponding stability estimate for the operators $\mathbf{I}+\mathbf{N}_r$. The latter is equivalent to showing that $I+\mathcal{N}T_r$ and hence that $T_r(I+\mathcal{N})T_r$ is stable on H_e^0 (cf. [6], Theorem 6).

Finally, we note that the stability of $T_r(I+\mathcal{N})T_r$ obviously follows from the stability of the finite section operators $T_r\phi_0(I+\widetilde{\mathcal{M}}_0)\phi_0T_r$ and $T_r\phi_1(I+\widetilde{\mathcal{M}}_1)\phi_1T_r$ on $L_2(0,1/2)$ and $L_2(1/2,1)$, respectively, and it remains to apply the stability result of §4.3(v), using (5.13). □

6 Error Analysis

In this section, we study the stability of the collocation method (3.14), (3.15) and give an error estimate in the L_2 norm. Using (5.1)-(5.3) and the collocation projection

$$\mathbf{P}_h = \begin{bmatrix} P_h & 0 \\ 0 & P_h \end{bmatrix},$$

with P_h defined in (4.20), Equations (3.14), (3.15), or equivalently (4.24), (4.25), can be written

$$\mathbf{P}_h\mathbf{B}\mathbf{z}_h = \mathbf{P}_h(\mathbf{A}+\mathbf{K})\mathbf{z}_h = \mathbf{P}_h\mathbf{f}, \ \mathbf{z}_h \in T_{e,h} \times T_{e,h}. \tag{6.1}$$

However, the stability can only be proved by allowing the possibility that the method be modified slightly, i.e. by cutting off around the corners at $x = 0$ and $x = 1$. Let T_{i^*h} be the truncation operator introduced in (5.16) with $r = i^*h$, and instead of (6.1), consider the modified collocation method

$$\mathbf{P}_h(\mathbf{A}+\mathbf{K}_{i^*h})\mathbf{z}_h = \mathbf{P}_h\mathbf{f}, \ \mathbf{z}_h \in T_{e,h} \times T_{e,h}, \tag{6.2}$$

where

$$\mathbf{K}_{i^*h} = \begin{bmatrix} (\mathcal{V}_{11}-\mathcal{A}_e)T_{i^*h} & 0 \\ \mathcal{K}_{21}T_{i^*h} & \mathcal{K}_{22} \end{bmatrix}. \tag{6.3}$$

Lemma 5.2 allows us to rewrite (5.1) as the (formally second-kind) equation

$$(\mathbf{I}+\mathbf{M})\mathbf{z} = \mathbf{e}, \ \text{with } \mathbf{M} = \mathbf{A}^{-1}\mathbf{K}, \ \mathbf{e} = \mathbf{A}^{-1}\mathbf{f}. \tag{6.4}$$

We now attack the stability of (6.2) by writing this method as a non-standard projection method for (6.4). For any $\mathbf{z} \in H_e^0 \times H_e^1$, let $\mathbf{R}_h\mathbf{z} \in T_{e,h} \times T_{e,h}$ solve the collocation equations

$$\mathbf{P}_h\mathbf{A}\mathbf{R}_h\mathbf{z} = \mathbf{P}_h\mathbf{A}\mathbf{z}. \tag{6.5}$$

The following lemma shows that $\mathbf{R}_h$ is a well defined projection operator with range $T_{e,h} \times T_{e,h}$.

Lemma 6.1 *For any $\mathbf{z} \in H_e^0 \times H_e^1$, the unique solution to (6.5) is given by*

$$\begin{aligned}
\mathbf{R}_h \mathbf{z} &= \begin{bmatrix} R_h & Q_h \\ 0 & P_h \end{bmatrix} \mathbf{z}, \\
R_h &= \mathcal{A}_e^{-1} P_h \mathcal{A}_e, \; Q_h = \mathcal{A}_e^{-1} P_h \mathcal{V}_{12}(I - P_h).
\end{aligned} \tag{6.6}$$

Moreover, for any $\mathbf{z} \in H_e^m \times H_e^m$, $m \geq 1$, we have the error estimate

$$||(\mathbf{I} - \mathbf{R}_h)\mathbf{z}||_{H_e^0 \times H_e^0} \leq ch^m ||\mathbf{z}||_{H_e^m \times H_e^m}. \tag{6.7}$$

Proof. Since P_h commutes with $\mathcal{A}_e$ on $T_{e,h}$, the unique solution to (6.5) is (cf. Lemma 5.2)

$$\mathbf{R}_h \mathbf{z} = \begin{bmatrix} \mathcal{A}_e^{-1} & -\mathcal{A}_e^{-1} P_h \mathcal{V}_{12} \\ 0 & P_h \end{bmatrix} \mathbf{P}_h \begin{bmatrix} \mathcal{A}_e & \mathcal{V}_{12} \\ 0 & I \end{bmatrix} \mathbf{z},$$

which gives (6.6). Moreover, using (4.21) we obtain

$$\begin{aligned}
||(\mathbf{I} - \mathbf{R}_h)\mathbf{z}||_{H_e^0 \times H_e^0} &\leq c\{||(I - R_h)z_1||_0 + ||(I - P_h)z_2||_0 + ||Q_h z_2||_0\} \\
&\leq c||(I - P_h)\mathcal{A}_e z_1||_1 + c||(I - P_h)z_2||_0 \\
&\leq ch^m\{||\mathcal{A}_e z_1||_{m+1} + ||z_2||_m\} \\
&\leq ch^m ||\mathbf{z}||_{H_e^m \times H_e^m}.
\end{aligned}$$

□

Using Lemma 6.1, it is easily seen that $\mathbf{z}_h$ solves (6.2) if and only if

$$\mathbf{z}_h + \mathbf{R}_h \mathbf{M}_{i^*h} \mathbf{z}_h = \mathbf{R}_h \mathbf{e}, \tag{6.8}$$

where (cf. (5.8), (5.9), (6.3))

$$\mathbf{M}_{i^*h} = \mathbf{A}^{-1} \mathbf{K}_{i^*h} = \begin{bmatrix} \mathcal{M} T_{i^*h} & \mathcal{E} \\ \mathcal{K}_{21} T_{i^*h} & \mathcal{K}_{22} \end{bmatrix}. \tag{6.9}$$

The following lemma is crucial for the stability of (6.8).

Lemma 6.2 *Assume $q \geq 2$. For each $\epsilon \geq 0$, there exists $i^* \geq 1$ independent of h such that*

$$||(\mathbf{I} - \mathbf{R}_h)\mathbf{M}_{i^*h}\mathbf{z}||_{H_e^0 \times H_e^0} \leq \epsilon ||\mathbf{z}||_{H_e^0 \times H_e^0}, \; \mathbf{z} \in H_e^0 \times H_e^0, \tag{6.10}$$

for all h sufficiently small.

Proof. From (6.6) and (6.9) we obtain for all $\mathbf{z} \in H_e^0 \times H_e^0$

$$\|(\mathbf{I} - \mathbf{R}_h)\mathbf{M}_{i^*h}\mathbf{z}\|_{H_e^0 \times H_e^0} \leq \|(I - R_h)\mathcal{M}T_{i^*h}z_1\|_0 + c\|(I - P_h)\mathcal{K}_{21}T_{i^*h}z_1\|_0 + \|(I - R_h)\mathcal{E}z_2\|_0 + c\|(I - P_h)\mathcal{K}_{22}z_2\|_0. \tag{6.11}$$

Here we have used the uniform boundedness of $\mathcal{A}_e^{-1}P_h\mathcal{V}_{12}$ on H_e^0 which is a consequence of estimate (4.21). Furthermore, since R_h converges strongly to the identity on H_e^0 and since $\mathcal{K}_{22}$ is a bounded operator of H_e^0 into H_e^1 for $q \geq 2$, we have

$$\|(I - R_h)\mathcal{E}z_2\|_0 + c\|(I - P_h)\mathcal{K}_{22}z_2\|_0 \leq \epsilon\|z_2\|_0 \tag{6.12}$$

for all sufficiently small h. To estimate the first two terms on the right side of (6.11), we observe that (4.21) (with $t = 0$, $s = 1$) and (6.7) imply the estimate

$$\|(I - P_h)z\|_0 + \|(I - R_h)z\|_0 \leq ch\|Dz\|_0, \ z \in H_e^1,$$

since $I - P_h$ and $I - R_h$ annihilate the constants. Together with (5.9) and (4.18), we then obtain for any $z \in H_e^0$

$$\begin{aligned} \|(I - R_h)\mathcal{M}T_{i^*h}z\|_0 + \|(I - P_h)\mathcal{K}_{21}T_{i^*h}z\|_0 &\leq ch\|D\mathcal{M}T_{i^*h}z\|_0 + ch\|D\mathcal{K}_{21}T_{i^*h}z\|_0 \\ &\leq ch\{\|D^2(\mathcal{V}_{11} - \mathcal{A}_e)T_{i^*h}z\|_0 + \|D^2\mathcal{V}_{12}\mathcal{K}_{21}T_{i^*h}z\|_0 \\ &\quad + \|D\mathcal{K}_{21}T_{i^*h}z\|_0\}. \end{aligned} \tag{6.13}$$

An inspection of the proofs of Lemmas 5.1 and 5.3 shows that, for $q \geq 2$, each of the operators $D(\mathcal{V}_{11} - \mathcal{A}_e)$, $D\mathcal{V}_{12}\mathcal{K}_{21}$ and $\mathcal{K}_{21}$ takes the form

$$\chi_0\widetilde{\mathcal{K}}_0\chi_0 + \mathcal{R}\chi_0\widetilde{\mathcal{K}}_1\chi_0\mathcal{R} + \mathcal{E}, \tag{6.14}$$

where $\widetilde{\mathcal{K}}_0$, $\widetilde{\mathcal{K}}_1$ are Mellin convolution operators on $\mathbb{R}^+$ with symbols of class $\Sigma_{-1,1}^{-\infty}$, $\mathcal{R}$ is the reflection operator defined by $(\mathcal{R}z)(x) = z(1 - x)$, and $\mathcal{E}$ is a bounded operator of H_e^0 into H_e^1.

We are now left with proving the following fact. Let $\mathcal{K}$ be a Mellin convolution operator of the form (4.26) with kernel function K and symbol $\widehat{K} \in \Sigma_{-1,1}^{-\infty}$. Then the estimate

$$\|D\mathcal{K}\phi_r v\|_0 \leq (c/r)\|v\|_0, \ v \in L_2(0,1), \ 0 < r < 1 \tag{6.15}$$

holds, where ϕ_r is the characteristic function of $(r, 1)$ and the constant c does not depend on v and r.

Indeed, combining the estimates (6.11)-(6.13) and applying (6.15) with $r = i^*h$ and i^* sufficiently large to the corresponding operators of the form (6.14) in (6.13), we obtain (6.10).

To prove (6.15), we observe that

$$\begin{aligned} |D\mathcal{K}\phi_r v(x)| &\leq \int_r^1 |D_x K(x/y)|y^{-1}|v(y)|dy \leq \int_r^1 |K'(x/y)|y^{-2}|v(y)|dy \\ &\leq r^{-1}\int_0^1 |K'(x/y)|y^{-1}|v(y)|dy, \ x \in (0,1). \end{aligned} \tag{6.16}$$

Since $\widehat{K} \in \Sigma_{-1,1}^{-\infty}$, the kernel estimates (4.28) (with $k = 1$ and $-1 < \rho < 1$) imply that the Mellin convolution kernel $|K'(x/y)|y^{-1}$ satisfies $x^{-1/2}|K'(x)| \in L_1(\mathbb{R}^+)$. Therefore, taking L_2 norms in (6.16) and applying §4.3(ii), gives the result. □

We are now in the position to prove our convergence result for the collocation method (6.2).

Theorem 6.3 *Assume (A1) and $q \geq 2$, and suppose that i^* is sufficiently large. Then, for all h sufficiently small and all $\mathbf{f} \in H_e^s \times H_e^s$, $s > 1/2$, there is a unique solution $\mathbf{z}_h \in T_{e,h} \times T_{e,h}$ of (6.2). Moreover, if for some $m \geq 1$ the exact solution $\mathbf{z}$ of (5.1) satisfies*

$$\mathbf{z} = [x(1-x)]^m \mathbf{v}, \text{ with } \mathbf{z} \in H_e^m \times H_e^m, \ \mathbf{v} \in H_e^0 \times H_e^0, \tag{6.17}$$

then we have the error estimate

$$\|\mathbf{z} - \mathbf{z}_h\|_{H_e^0 \times H_e^0} \leq ch^m (\|\mathbf{z}\|_{H_e^m \times H_e^m} + \|\mathbf{v}\|_{H_e^0 \times H_e^0}). \tag{6.18}$$

Remark 3 *It can be proved that the solution $\mathbf{z}$ of (5.1) takes the form (6.17) with arbitrarily large m if the functions f and g in (1.1) are sufficiently smooth and the grading exponent in (2.1) is large enough: see (1.6).*

Proof of Theorem 6.3. First, from Lemmas 6.2 and 5.6, we immediately obtain the stability of the equivalent method (6.8), i.e. the estimate

$$\|(\mathbf{I} + \mathbf{R}_h \mathbf{M}_{i^*h})\mathbf{z}_h\|_{H_e^0 \times H_e^0} \geq c\|\mathbf{z}_h\|_{H_e^0 \times H_e^0}, \ \mathbf{z}_h \in T_{e,h} \times T_{e,h}, \tag{6.19}$$

as $h \to 0$ whenever i^* is sufficiently large. This gives the first assertion since the right side of (6.2) is well defined for $\mathbf{f} \in H_e^s \times H_e^s$, $s > 1/2$.

To prove the error estimate (6.18), we note that

$$\|\mathbf{z} - \mathbf{z}_h\|_{H_e^0 \times H_e^0} \leq \|(\mathbf{I} - \mathbf{P}_h)\mathbf{z}\|_{H_e^0 \times H_e^0} + \|\mathbf{z}_h - \mathbf{P}_h \mathbf{z}\|_{H_e^0 \times H_e^0},$$

where the first term is of order h^m by (4.21) and (6.17).

Furthermore, using (6.19) and then (6.8) with (6.4) and (6.7), we obtain

$$\begin{aligned} \|\mathbf{z}_h - \mathbf{P}_h \mathbf{z}\|_{H_e^0 \times H_e^0} &\leq c\|(\mathbf{I} + \mathbf{R}_h \mathbf{M}_{i^*h})(\ \mathbf{z}_h - \mathbf{P}_h \mathbf{z})\|_{H_e^0 \times H_e^0} \\ &= c\|\mathbf{R}_h[(\mathbf{I} + \mathbf{M})\mathbf{z} - (\mathbf{I} + \mathbf{M}_{i^*h})\mathbf{P}_h \mathbf{z}]\|_{H_e^0 \times H_e^0} \\ &\leq c\|(\mathbf{R}_h - \mathbf{P}_h)\mathbf{z}\|_{H_e^0 \times H_e^0} + c\|\mathbf{R}_h \mathbf{M}\mathbf{z} - \mathbf{R}_h \mathbf{M}_{i^*h} \mathbf{P}_h \mathbf{z}\|_{H_e^0 \times H_e^0} \\ &\leq ch^m \|\mathbf{z}\|_{H_e^m \times H_e^m} + c\|\mathbf{M}\mathbf{z} - \mathbf{M}_{i^*h} \mathbf{P}_h \mathbf{z}\|_{H_e^0 \times H_e^0} \\ &\quad + ch\|D(\mathbf{M}\mathbf{z} - \mathbf{M}_{i^*h} \mathbf{P}_h \mathbf{z})\|_{H_e^0 \times H_e^0}. \end{aligned} \tag{6.20}$$

Using (4.21) and (6.17), the second term on the right side of (6.20) can now be estimated by

$$\begin{aligned} \|\mathbf{M}\mathbf{z} - \mathbf{M}_{i^*h} \mathbf{P}_h \mathbf{z}\|_{H_e^0 \times H_e^0} &\leq \|\mathbf{M}_{i^*h}(\ \mathbf{I} - \mathbf{P}_h)\mathbf{z}\|_{H_e^0 \times H_e^0} + \|(\mathbf{M} - \mathbf{M}_{i^*h})\mathbf{z}\|_{H_e^0 \times H_e^0} \\ &\leq c\|(\mathbf{I} - \mathbf{P}_h)\mathbf{z}\|_{H_e^0 \times H_e^0} + c\|(I - T_{i^*h})z_1\|_0 \\ &\leq ch^m (\|\mathbf{z}\|_{H_e^m \times H_e^m} + \|\mathbf{v}\|_{H_e^0 \times H_e^0}). \end{aligned}$$

Note that $\mathbf{M}_{i^*h}$ is uniformly bounded since $\mathbf{M}$ is bounded on $H_e^0 \times H_e^0$ (see §5) and the truncation operator T_{i^*h} is uniformly bounded on H_e^0.

It remains to show that the last term of (6.20) is of order h^m. Using (6.9) and (4.21), we have

$$\begin{aligned}
\|D(\mathbf{Mz} - \mathbf{M}_{i^*h}\mathbf{P}_h\mathbf{z})\|_{H_e^0 \times H_e^0} &\le \|D(\mathcal{M} - \mathcal{M}T_{i^*h}P_h)z_1\|_0 + \|D(\mathcal{K}_{21} - \mathcal{K}_{21}T_{i^*h}P_h)z_1\|_0 \\
&\quad + \|D\mathcal{E}(I-P_h)z_2\|_0 + \|D\mathcal{K}_{22}(I-P_h)z_2\|_0 \\
&\le ch^m\|z_2\|_m + \|D\mathcal{M}T_{i^*h}(I-P_h)z_1\|_0 + \|D\mathcal{K}_{21}T_{i^*h}(I-P_h)z_1\|_0 \\
&\quad + \|D\mathcal{M}(I-T_{i^*h})z_1\|_0 + \|D\mathcal{K}_{21}(I-T_{i^*h})z_1\|_0.
\end{aligned}$$

From the proof of Lemma 6.2 we see that the second and the third term can be bounded by

$$(c/i^*h)\|(I-P_h)z_1\|_0 \le ch^{m-1}\|z_1\|_m.$$

To estimate the last two terms, we again proceed as in Lemma 6.2 and are left with proving the estimate

$$\|D\mathcal{K}\psi_r x^m v\|_0 \le cr^{m-1}\|v\|_0, \ v \in L_2(0,1), \ 0 < r < 1,$$

where ψ_r is the characteristic function of $(0,r)$ and $\mathcal{K}$ is a Mellin convolution operator with kernel K and symbol of class $\Sigma_{-1,1}^{-\infty}$. We have

$$\begin{aligned}
|D\mathcal{K}\psi_r x^m v(x)| &\le \int_0^r |D_x K(x/y)| y^{m-1}|v(y)|dy \\
&= \int_0^r |K'(x/y)|y^{-1}|y^{m-1}v(y)|dy \\
&\le r^{m-1}\int_0^1 |K'(x/y)|y^{-1}|v(y)|dy, \ x \in (0,1),
\end{aligned}$$

and as in the proof of (6.15) we obtain the result. □

Acknowledgments

The work of Y. Jeon was carried out while he was a Research Associate in the School of Mathematics of the University of New South Wales. The work of E. P. Stephan was carried out while he was Visiting Professor in the School of Mathematics of the University of New South Wales. The work of J. Elschner began while he was visiting the University of New South Wales and the University of Queensland. The support of the Australian Research Council is gratefully acknowledged by all authors.

References

[1] K.E. Atkinson and I.H. Sloan(1991), The numerical solution of first-kind logarithmic-kernel integral equations on smooth open arcs, *Math. Comp.*, **56**, pp. 119-139

[2] G.A. Chandler and I.G. Graham(1988), Product integration collocation methods for non-compact integral operator equations, *Math. Comp.*, **50**, pp. 125-138

[3] M. Costabel and E.P. Stephan(1985), Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation, Mathematical Models and Methods in Mechanics, Banach Center Publications, **15**, PWN-Polish Scientific Publishers, Warsaw, pp. 175-251

[4] M. Costabel and E.P. Stephan(1987), On the convergence of collocation methods for boundary integral equations on polygons, *Math. Comp.*, **49**, pp. 461-478

[5] J. Elschner(1990), On spline approximation for a class of non-compact integral equations, *Math. Nachr.*, **146**, pp. 271-321

[6] J. Elschner and I.G. Graham(1995), An optimal order collocation method for first kind boundary integral equations on polygons, *Numer. Math.*, **70**, pp. 1-31

[7] G.I. Eskin(1981), *Boundary Value Problems for Elliptic Pseudodifferential Equations*, American Mathematical Society, Providence

[8] I. Gohberg and I.A. Feldman(1974), *Convolution Equations and Projection Methods for their Solution*, American Mathematical Society, Providence

[9] Y. Jeon(1993), A Nyström method for boundary integral equations on domains with piecewise smooth boundary, *J. Integral Eqns Appl.*, **5**, pp. 221-242

[10] Y. Jeon(1994), A quadrature method for the Cauchy singular integral equations, submitted

[11] R. Kress(1990), A Nyström method for boundary integral equations in domains with corners, *Numer. Math.*, **58**, pp. 145-161

[12] S.G. Mikhlin and S. Prössdorf(1986), *Singular Integral Operators*, Springer-Verlag, Berlin

[13] Y. Yan and I.H. Sloan(1988), On integral equations of the first kind with logarithmic kernels, *J. Integral Eqns Appl.*, **1**, pp. 549-579